Journal of Computational Physics 228 (2009) 4970–4975

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Short Note

Alleviation of cancellation problem of preconditioned Navier–Stokes equations

Sang-Hyeon Lee *

Department of Aerospace Engineering, University of Ulsan, Mugeo-Dong, Nam-Gu, Ulsan 680-749, Republic of Korea

ARTICLE INFO

Article history:
Received 28 September 2008
Received in revised form 13 April 2009
Accepted 16 April 2009
Available online 24 April 2009

Keywords:
Cancellation problem
Preconditioning
Navier–Stokes equations

ABSTRACT

An approach to alleviate the cancellation problem of preconditioned Navier-Stokes equations is proposed. Adiabatic laminar viscous flows around a circular cylinder are calculated at different Mach numbers. It is shown that a redefinition of total enthalpy to reduce the magnitude of total enthalpy can alleviate the cancellation problem for adiabatic laminar flows at low Mach numbers.

1. Introduction

Preconditioning methods have received growing attention due to their various applications in CFD fields. The preconditioning method pre-multiplies the time derivative by a suitable preconditioner that scales the eigenvalues to the same order of magnitude [1]. A number of preconditioners have been suggested to solve the stiffness problem [2–7]. Choi and Merkle [5] suggested a preconditioner that introduced well-conditioned eigenvalues and has been extended for use in many CFD applications [6–13].

However, Sabanca et al. [12] and Lee [14] reported that preconditioned Euler equations had serious convergence problems at very low Mach numbers. This is attributed to cancellation errors occurring due to accumulation or magnification of round-off errors. Round-off errors are mainly determined by the precision of floating-point variables and thus are inevitable. However, the cancellation errors can be minimized by adopting an efficient algorithm. There are two main sources of cancellation errors: (1) spatial discretization and (2) preconditioner. Sesterhenn et al. [15] focused their attention on the errors related to the spatial discretization. Lee [16] analyzed the behavior of the preconditioner and showed that the cancellation problem is strongly related to the element including total enthalpy in the preconditioner but did not suggest possible ways to avoid the cancellation problem. In the present study, an approach to alleviate the cancellation problem of preconditioned Navier–Stokes equations is proposed and verified with numerical calculations. Adiabatic laminar compressible flows around a circular cylinder are calculated at different low Mach numbers.

2. Cancellation errors

2.1. Governing equations

The non-dimensionalized governing equations considered in the present study are the two-dimensional preconditioned Navier–Stokes equations. The preconditioner, Γ, considered in the present study is Choi and Merkle's preconditioner.

* Tel.: +82 52 259 2825; fax: +82 52 259 1681.
E-mail address: lsh@mail.ulsan.ac.kr

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2009.04.024

$$\Gamma\frac{\partial Q}{\partial t}+\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}=\frac{\partial E_v}{\partial x}+\frac{\partial F_v}{\partial y} \tag{2.1.1a}$$

$$Q=\begin{bmatrix}p\\u\\v\\T\end{bmatrix},\quad \begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho u v\\ \rho h_o u\end{bmatrix},\quad F=\begin{bmatrix}\rho v\\ \rho u v\\ \rho v^2+p\\ \rho h_o v\end{bmatrix},\quad E_v=\frac{1}{Re_c}\begin{bmatrix}0\\ \tau_{xx}\\ \tau_{xy}\\ u_j\tau_{xj}-q_x\end{bmatrix},$$

$$F_v=\frac{1}{Re_c}\begin{bmatrix}0\\ \tau_{yx}\\ \tau_{yy}\\ u_j\tau_{yj}-q_y\end{bmatrix},\quad \Gamma=\begin{bmatrix}\frac{1}{\beta} & 0 & 0 & 0\\ \frac{u}{\beta} & \rho & 0 & 0\\ \frac{v}{\beta} & 0 & \rho & 0\\ \frac{h_o}{\beta}-1 & \rho u & \rho v & \rho c_p\end{bmatrix} \tag{2.1.1b}$$

$$p=\rho RT \tag{2.1.1c}$$

$$\beta=c^2M_r^2,\quad M_r^2=\min\left[1,\max\left(M^2,0.25M_\infty^2\right)\right] \tag{2.1.1d}$$

The term q_k is the heat transfer rate in the k-direction. The term Re_c is the Reynolds number based on the speed of sound.

The governing equations are non-dimensionalized with the quantities at the infinite far field: $\gamma_\infty p_\infty$ (pressure), ρ_∞ (density), $\gamma_\infty T_\infty$ (temperature), c_∞ (speed of sound), R_∞ (gas constant), γ_∞ (specific heats ratio), μ_∞ (viscosity) and L (characteristic length). The non-dimensionalized quantities have the following orders of magnitude.

$$u,v\sim O(M),\quad p,\rho,T\sim O(1),\quad R,c_p\sim O(1) \tag{2.1.2}$$

In the present study, the adiabatic laminar compressible flows are considered. Then the changes of the thermodynamic variables have the following orders of magnitude.

$$p,\rho,T\sim O(M_\infty^2) \tag{2.1.3}$$

Lee [16] showed that pre-multiplying the system of equations by the inverse of the preconditioner resulted in better convergence characteristics at low Mach numbers. Thus, the following form of governing equations is considered in the present study.

$$\frac{\partial Q}{\partial t}+\Gamma^{-1}\left(\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}\right)=\Gamma^{-1}\left(\frac{\partial E_v}{\partial x}+\frac{\partial F_v}{\partial y}\right) \tag{2.1.4a}$$

$$\Gamma^{-1}=\begin{bmatrix}\beta & 0 & 0 & 0\\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0\\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0\\ \frac{\beta-h_0+u^2+v^2}{\rho c_p} & -\frac{u}{\rho c_p} & -\frac{v}{\rho c_p} & \frac{1}{\rho c_p}\end{bmatrix}. \tag{2.1.4b}$$

Lee [16] reported that the magnification of cancellation error is strongly related to magnitude of the term h_o/β-1 in the preconditioner. This suggests that the cancellation problem can be alleviated if we control the magnitude of the term h_o/β.

The total enthalpy of a non-reacting ideal pure gas is defined as follows.

$$h_o=h(T)+\frac{u^2+v^2}{2}=\int_{T=T_S}^{T}c_p(T)dT+\frac{u^2+v^2}{2} \tag{2.1.5}$$

The term T_S is the temperature at a standard state. For a non-reacting pure gas, the difference of enthalpy is of importance. Thus, we can take any temperature for the standard temperature. Three definitions of enthalpy according to different standard temperatures are considered in the present study.

$$h_o=\int_0^T c_p(T)dT+\frac{u^2+v^2}{2},\quad T_S=0,\quad \text{definition HA} \tag{2.1.6a}$$

$$h_o=\int_{T_{298}}^T c_p(T)dT+\frac{u^2+v^2}{2},\quad T_S=T_{298}=298/\gamma_\infty T_\infty,\quad \text{definition HB} \tag{2.1.6b}$$

$$h_o=\int_{T_{\inf}}^T c_p(T)dT+\frac{u^2+v^2}{2},\quad T_S=T_{\inf}=T_\infty/\gamma_\infty T_\infty=1/\gamma_\infty,\quad \text{definition HC} \tag{2.1.6c}$$

The different definitions of the enthalpy result in the different magnitudes of the enthalpy. For definition HA, the total enthalpy is $O(1)$. Also, for definition HB, the total enthalpy is $O(1)$ in case T_{298} is different from the inflow temperature, $T_{\inf}$. However, for definition HC, the total enthalpy is $O(M_\infty^2)$ since the temperature change in adiabatic flows is due to the change of the flow velocity.

2.2. Error analysis

The numerical operator that expressed round-off errors during arithmetic operations is defined as follows.

$$\langle a * b \rangle = (a * b)(1 + \varepsilon) \tag{2.2.1}$$

The operator "*" denotes one of the floating-point arithmetic operators. The error is estimated as $|\varepsilon| \leqslant 5 \times 10^{-d}$ when d decimals are available for representation of the mantissa. Let the round-off error, ε, be sufficiently small.

In the present study, low Mach number flows will be considered, since the effects of cancellation errors appear at low Mach numbers. The explicit calculations are considered for simplicity. The explicit form of the governing equations can be expressed as follows:

$$\Delta Q = \Gamma^{-1} RES \tag{2.2.2a}$$

$$RES = [R_p, R_u, R_v, R_T]^T - \Delta t \left[\frac{\Delta(E - E_v)}{\Delta x} - \frac{\Delta(F - F_v)}{\Delta y} \right] \tag{2.2.2b}$$

Ideal or error-free solutions are assumed and symbolized by Δp^*, Δu^*, ΔT^*. The solution change, ΔQ, can be expressed in the following form.

$$\Delta p = \langle \beta R_P \rangle = \beta R_p (1 + \varepsilon) = \Delta p^* (1 + \varepsilon) \tag{2.2.3a}$$

$$\Delta u = \left\langle \frac{1}{\rho} \langle R_u - \langle u R_p \rangle \rangle \right\rangle \approx \Delta u^* - \frac{u}{\rho\beta} \Delta p^* \varepsilon \tag{2.2.3b}$$

$$\begin{aligned} \Delta T &= \left\langle \frac{1}{\rho c_p} \langle R_T - \langle \langle h_o - \beta - u^2 - v^2 \rangle R_p \rangle - \langle u R_u \rangle - \langle v R_v \rangle \rangle \right\rangle \\ &\approx \Delta T^* - \frac{h_o - \beta - u^2 - v^2}{\rho c_p \beta} 2 \Delta p^* \varepsilon \end{aligned} \tag{2.2.3c}$$

Refer to Lee [16] for details.

The preconditioning parameter, β, at a low Mach number can be expressed as follows.

$$\beta \sim M_r^2 \tag{2.2.4}$$

Then, the errors can be expressed in the following forms.

$$\Delta p - \Delta p^* \sim \varepsilon \tag{2.2.5a}$$

$$\Delta u - \Delta u^* \sim \varepsilon \frac{\Delta p^*}{M_r^2} M \tag{2.2.5b}$$

$$\Delta T - \Delta T^* \sim \varepsilon \frac{\Delta p^*}{M_r^2} \quad \text{according to definition HA or HB} \tag{2.2.5c}$$

$$\Delta T - \Delta T^* \sim \varepsilon \frac{\Delta p^*}{M_r^2} M^2 \sim \varepsilon \Delta p^* \quad \text{according to definition HC} \tag{2.2.5d}$$

It should be noted that the temperature error according to definition HA or HB is much larger than the velocity error at low Mach numbers. This suggests that the temperature field could not be resolved even if the velocity field is resolved at a very low Mach number. On the other hand, the temperature error according to definition HC is much smaller than the velocity error. Thus, the modification of the definition of enthalpy as Eq. (2.1.6c) could alleviate the cancellation problem of the energy equation.

3. Numerical methods

3.1. Discretization

A finite volume method is used to discretize the preconditioned governing equations. In order to get the flux vector at the surface of a grid cell, the preconditioned Roe's FDS (Flux Difference Splitting) scheme [17] with the third-order spatial accuracy is used. The van Albada limiter [18] is used to avoid numerical oscillations. The preconditioned LU-SGS (Lower Upper Symmetric Gauss Seidel) scheme [9,10] is used for time integration.

3.2. Grid system and flow conditions

The calculations were conducted on an O-type grid system. The radius of the outer boundary is 50 times that of the cylinder diameter. The number of grid points is 7200 (120×60). The grid points in the circumferential direction are evenly distributed and the grid points in the radial direction are clustered towards the cylinder wall up to where the aspect ratio of the nearest grid point from the wall is about unity.

The working fluid is an ideal gas. The inflow pressure and temperature are 1.0 atm and 300 K, respectively. The inflow and outflow boundaries are specified with the characteristic boundary conditions [1,19,20]. As suggested by Okong'o and Bellan [20], the density and velocity at the inflow boundary are fixed, and the pressure is updated according to the outgoing wave amplitude variations determined from interior points. The pressure at the outflow is fixed, and the remaining wave amplitude variations are determined from the interior points. The no-slip boundary condition is applied at the solid walls. On adiabatic walls, the temperature, density and pressure are determined to be the same as those at the nearest grid point.

3.3. Algorithms and precisions

The calculations were conducted using double precision variables that store 15 significant digits. The concept of relative treatments of all the variables and flux vectors, suggested by Sesterhenn et al. [15], is adopted to reduce the loss of significant figures.

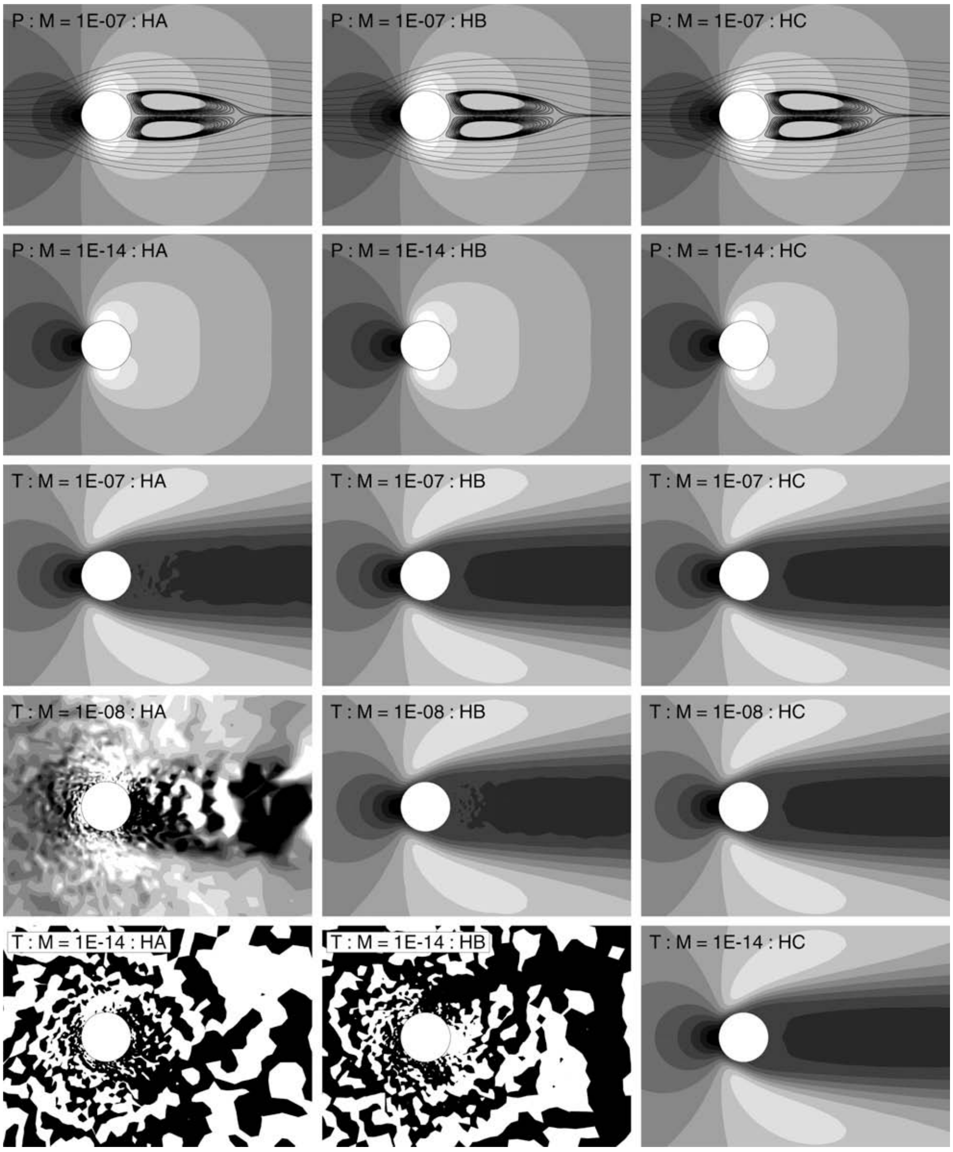

Fig. 1. Pressure and temperature fields around circular cylinder at different Mach numbers. $Re = 40$.

3.4. Estimation of convergence

The residual decays of velocity, pressure and temperature are plotted to represent the convergence characteristics. Lee [14,16] showed that it was necessary to renormalize the residuals in order to represent the convergence characteristics as follows.

$$RES(p) = \frac{\sum_{i,j}|\Delta p|}{N_G M_\infty^2}, \quad RES(u,v) = \frac{\sum_{i,j}\sum_{q=u,v}|\Delta q|}{N_G M_\infty}, \quad RES(T) = \frac{\sum_{i,j}|\Delta T|}{N_G M_\infty^2} \tag{3.4.1}$$

4. Results

Fig. 1 shows the pressure and temperature fields around a circular cylinder at very low Mach numbers. The Reynolds number is 40. The pressure fields even at extremely low Mach numbers can be resolved regardless of the definitions of enthalpy. However, the definition of enthalpy has strong influences on the temperature fields at low Mach numbers. For def-

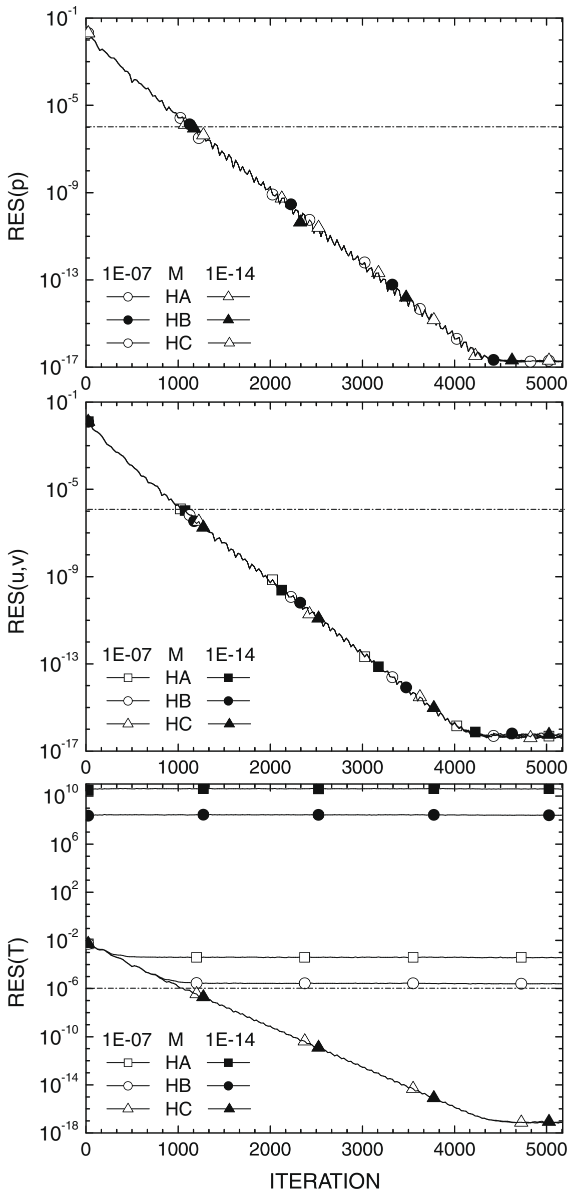

Fig. 2. Convergence histories of pressure, velocity and temperature at different Mach numbers.

inition HA, there are unphysical wiggle patterns in wake regions at $M = 10^{-7}$, moreover, any meaningful features of the temperature fields are not found at $M = 10^{-8}$. For definition HB, the temperature fields are well resolved at $M = 10^{-7}$ but unphysical wiggle patterns in the temperature fields begin to appear at $M = 10^{-8}$. The temperature fields for definitions HA and HB show chaotic features at $M = 10^{-14}$. For definition HC, the temperature fields are well resolved even at $M = 10^{-14}$.

Fig. 2 compares the convergence characteristics of the methods adopting definition HA, HB and HC at $M = 10^{-7}$ and 10^{-14}. The horizontal guidelines indicate the critical renormalized residuals required for enough convergence. The convergence histories of the continuity and momentum equations according to definitions HA, HB and HC are exactly same regardless of Mach numbers. However, the definition of enthalpy has strong influences on the temperature fields at low Mach numbers. For definition HA, the temperature residual do not converge below the guideline even at $M = 10^{-7}$. For definition HB, the temperature residual approaches near to the guideline at $M = 10^{-7}$but do not converge at $M = 10^{-14}$. However, for definition HC, the temperature residuals converge below the guideline even at $M = 10^{-14}$. The difference between the standard temperatures for definitions HB and HC is 2 K, but the convergence characteristics according to definitions HB and HC are remarkably different especially at extremely low Mach numbers.

5. Concluding remarks

The cancellation problem of the preconditioned Navier–Stokes equations is analyzed and an approach to alleviate the cancellation problem for adiabatic low Mach number flows is suggested. The convergence characteristics of the energy equation are strongly affected by the definition of enthalpy, and also a proper definition of enthalpy alleviates the cancellation problem of the energy equation of the preconditioned Navier–Stokes equations.

Acknowledgement

This work was supported by the Research Fund from University of Ulsan.

References

[1] E. Turkel, Annu. Rev. Fluid Mech. 31 (1999) 385.
[2] E. Turkel, J. Comput. Phys. 72 (1987) 277.
[3] E. Turkel, Appl. Numer. Math. 12 (1993) 257.
[4] B. van Leer, W-T. Lee, P. Roe, AIAA paper 91-1552-CP, 1991.
[5] Y.H. Choi, C.L. Merkle, J. Comput. Phys. 105 (1993) 207.
[6] J.M. Weiss, W.A. Smith, AIAA J. 33 (1995) 2050.
[7] C.J. Merkle, AIAA J. 36 (1998) 515.
[8] J.S. Shuen, K.H. Chen, J. Comput. Phys. 106 (1993) 306.
[9] K.H. Chen, J.S. Shuen, E. Mularz, AIAA paper 95-0800, 1995.
[10] L.D. Dailey, R.H. Pletcher, AIAA paper 95-1668, 1995.
[11] S. Hsieh, V. Yang, Int. J. CFD 8 (1997) 31.
[12] M. Sabanca, G. Brenner, N. Alemdaroglu, Int. J. Numer. Meth. Fluids 34 (2000) 167.
[13] M. Sabanca, F. Durst, Phys. Fluids 15 (2003) 1821.
[14] S-H. Lee, J. Comput. Phys. 208 (2005) 266.
[15] J. Sesterhenn, B. Muller, H. Thomann, J. Comput. Phys. 151 (1999) 597.
[16] S-H. Lee, J. Comput. Phys. 225 (2007) 1199.
[17] S. Guillard, C. Viozat, Comput. Fluid 28 (1999) 63.
[18] P. Arminjon, A. St-Cyr, A. Madrane, Appl. Numer. Math. 40 (2002) 367.
[19] D.L. Darmofal, P. Moinier, M.B. Giles, J. Comput. Phys. 160 (2000) 369.
[20] N. Okong'o, J. Bellan, J. Comput. Phys. 176 (2002) 330.